

## Chapter Three: Orthogonal functions and Fourier series

This is going to be a short section. We just need to have a brief discussion about a couple of ideas that we will be dealing with on occasion as we move into the next topic of this chapter.

### 3.1 Periodic Function

The first topic we need to discuss is that of a periodic function. A function is said to be periodic with period  $T$  if the following is true,

$$f(x + T) = f(x) \quad \text{for all } x$$

Fact: If  $f$  and  $g$  are both periodic functions with period  $T$  then so is  $f + g$  and  $fg$ .

This is easy enough to prove so let's do that.

Since  $f$  and  $g$  are periodic functions by definition of periodic function we have

$$f(x + T) = f(x) \quad \text{and} \quad g(x + T) = g(x)$$

Now to prove that above:

$$(f + g)(x + T) = f(x + T) + g(x + T) = f(x) + g(x)$$

$$(fg)(x + T) = f(x + T)g(x + T) = f(x)g(x)$$

The two periodic functions that most of us are familiar are sine and cosine and in fact we'll be using these two functions regularly in the remaining sections of this chapter. So, having said that let's close off this discussion of periodic functions with the following fact,

$\cos(\omega x)$  and  $\sin(\omega x)$  are periodic functions of period  $T = \frac{2\pi}{\omega}$

### 3.2 Even and Odd Functions

The next quick idea that we need to discuss is that of even and odd functions.

Recall that a function is said to be even if,  $f(x) = f(-x)$

And a function is said to be odd if,  $f(-x) = -f(x)$

The standard examples of even functions are  $f(x) = x^2$  and  $g(x) = \cos(x)$  while the standard examples of odd functions are  $f(x) = x^3$  and  $g(x) = \sin(x)$ .

The following fact about certain integrals of even/odd functions will be useful in some of our work.

1) If  $f(x)$  is an even function then,

$$\int_{-L}^L f(x)dx = 2 \int_0^L f(x)dx$$

2) If  $f(x)$  is an odd function then,

$$\int_{-L}^L f(x)dx = 0$$

Note that this fact is only valid on a “symmetric” interval, *i.e.* an interval in the form  $[-L, L]$ .

If we are not integrating on a “symmetric” interval then the fact may or may not be true.

### 3.3 Orthogonal Functions

The final topic that we need to discuss here is that of orthogonal functions. This idea will be integral to what we’ll be doing in the remainder of this chapter and in the next chapter as we discuss one of the basic solution methods for partial differential equations.

Let’s first get the definition of orthogonal functions out of the way.

Definition:

- 1) Two non-zero functions  $f(x)$  and  $g(x)$  are said to be orthogonal on  $a \leq x \leq b$

$$\text{if, } \int_a^b f(x)g(x)dx = 0$$

- 2) A set of non-zero functions,  $\{f_i(x)\}$ , is said to be mutually orthogonal on  $a \leq x \leq b$  (or just an orthogonal set if we're being lazy) if  $f_i(x)$  and  $f_j(x)$  are orthogonal for every  $i \neq j$ . In other words,

$$\int_a^b f_i(x)f_j(x)dx = \begin{cases} 0 & i \neq j \\ c > 0 & i = j \end{cases}$$

Note that in the case of  $i = j$  for the second definition we know that we'll get a positive

value from the integral because,  $\int_a^b f_i(x)f_i(x)dx = \int_a^b [f_i(x)]^2 dx > 0$ .

Recall that when we integrate a positive function we know the result will be positive as well.

Also note that the non-zero requirement is important because otherwise the integral would be trivially zero regardless of the other function we were using.

Before we work some examples there are a nice set of trig formulas that we'll need to help us with some of the integrals.

$$\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$

$$\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

**Example 1:** Show that  $\left\{ \cos\left(\frac{n\pi x}{L}\right) \right\}_{n=0}^{\infty}$  is mutually orthogonal on  $-L \leq x \leq L$ .

Solution: This is not too difficult to do. All we really need to do is evaluate the following integral.

$$\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx$$

Before we start evaluating this integral let's notice that the integrand is the product of two even functions and so must also be even. This means that we can use Fact 3 above to write the integral as,

$$\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 2 \int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx$$

There are two reasons for doing this. First having a limit of zero will often make the evaluation step a little easier and that will be the case here. We will discuss the second reason after we're done with the example.

Now, in order to do this integral we'll actually need to consider three cases

*Case 1:*  $n = m = 0$

In this case the integral is very easy and is,

$$\int_{-L}^L \cos(0) \cos(0) dx = \int_{-L}^L dx = 2 \int_0^L dx = 2L$$

*Case 2:*  $n = m \neq 0$

This integral is a little harder than the first case, but not by much (provided we recall a simple trigonometric formula). The integral for this case is,

$$\begin{aligned}
\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx &= \int_{-L}^L \cos^2\left(\frac{n\pi x}{L}\right) dx = 2 \int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx \\
&= \int_0^L 1 + \cos\left(\frac{2n\pi x}{L}\right) dx \\
&= \left( x + \frac{1}{2n\pi} \sin\left(\frac{2n\pi x}{L}\right) \right) \Big|_0^L = L + \frac{L}{2n\pi} \sin(2n\pi)
\end{aligned}$$

Now, at the point we need to recall that  $n$  is an integer and so  $\sin(2n\pi) = 0$  and our final value for the is,

$$\int_{-L}^L \cos^2\left(\frac{n\pi x}{L}\right) dx = 2 \int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx = L$$

The first two cases are really just showing that if  $n = m$  the integral is not zero (as it shouldn't be) and depending upon the value of  $n$  (and hence  $m$ ) we get different values of the integral. Now we need to do the third case and this, in some ways, is the important case since we must get zero out of this integral in order to know that the set is an orthogonal set. So, let's take care of the final case.

*Case 3:  $n \neq m$*

This integral is the “messiest” of the three that we've had to do here. Let's just start off by writing the integral down.

$$\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 2 \int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx$$

In this case we can't combine/simplify as we did in the previous two cases. We can however, acknowledge that we've got a product of two cosines with different arguments and so we can use one of the trig formulas above to break up the product as follows,

$$\begin{aligned}
\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx &= 2 \int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx \\
&= \int_0^L \cos\left(\frac{(n-m)\pi x}{L}\right) + \cos\left(\frac{(n+m)\pi x}{L}\right) dx \\
&= \frac{L}{(n-m)\pi} \sin((n-m)\pi) + \frac{L}{(n+m)\pi} \sin((n+m)\pi)
\end{aligned}$$

Now, we know that  $n$  and  $m$  are both integers and so  $n-m$  and  $n+m$  are also integers and so both of the sines above must be zero and all together we get,

$$\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 2 \int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0$$

So, we've shown that if  $n \neq m$  the integral is zero and if  $n = m$  the value of the integral is a positive constant and so the set is mutually orthogonal.

**Example 2:** Show that  $\left\{ \sin\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty}$  is mutually orthogonal on  $-L \leq x \leq L$ .

Let's leave this for the reader as an exercise!!! Do not jump this one since we use it repeatedly.

### 3.4 Fourier sine series and Fourier cosine series

#### 3.4.1 Fourier Sine series

In this section we are going to start taking a look at Fourier series. We should point out that this is a subject that can span a whole class and what we'll be doing in this section (as well as the next couple of sections) is intended to be nothing more than a very brief look at the subject. The point here is to do just enough to allow us to do some basic solutions to partial differential

equations in the next chapter. There are many topics in the study of Fourier series that we'll not even touch upon here.

So, with that out of the way let's get started, although we're not going to start off with Fourier series. Let's instead think back to our Calculus class where we looked at Taylor Series. With Taylor Series we wrote a series representation of a function  $f(x)$ , as a series whose terms were powers of  $x - a$  for some  $x = a$ . with some conditions we were able to show that,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x-a)^n$$

and that the series will converge to  $f(x)$  on  $|x-a| < R$  for some  $R$  that will be dependent upon the function itself.

There is nothing wrong with this, but it does require that derivatives of all orders exist at  $x = a$ . Or in other words  $f^n(a)$  exists for  $n = 0, 1, 2, 3, 4, \dots$ . Also for some functions the value of  $R$  may end up being quite small.

These two issues (along with a couple of others) mean that this is not always the best way or writing a series representation for a function. In many cases it works fine and there will be no reason to need a different kind of series. There are times however where another type of series is either preferable or required.

We are going to build up an alternative series representation for a function over the course of the next couple of sections. The ultimate goal for the rest of this chapter will be to write down a series representation for a function in terms of sines and cosines.

We will start things off by assuming that the function,  $f(x)$ , we want to write a series representation for is an odd function i.e.  $f(-x) = -f(x)$ . Because  $f(x)$  is odd it makes some sense that should be able to write a series representation for this in terms of sines only (since they are also odd functions).

What we will try to do here is writing  $f(x)$  as the following series representation, called a *Fourier sine series*, on  $-L \leq x \leq L$ .

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{Where } B_n \text{ is called the coefficients.}$$

There are a couple of issues to note here. First, at this point, we are going to assume that the series representation will converge to  $f(x)$  on  $-L \leq x \leq L$ . Assuming that the series does converge to  $f(x)$  it is interesting to note that, unlike Taylor Series, this representation will always converge on the same interval and that the interval does not depend upon the function.

The question now is how to determine the coefficients  $B_n$ , in the series.

Let's start with the series above and multiply both sides by  $\sin\left(\frac{m\pi x}{L}\right)$  where  $m$  is a fixed integer in the range  $\{1, 2, 3, 4, \dots\}$ . In other words we multiply both sides by any of the sines in the set of sines that we are working with here. Doing this gives,

$$f(x) \sin\left(\frac{m\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right)$$

Now, let's integrate both sides of this from  $x = -L$  to  $x = L$ .

$$\int_{-L}^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = \int_{-L}^L \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

At this point we've got a small issue to deal with. We know from Calculus that an integral of a finite series (more commonly called a finite sum....) is nothing more than the (finite) sum of the integrals of the pieces. In other words for finite series we can interchange an integral and a series. For infinite series however, we cannot always do this. For some integrals of infinite series we cannot interchange an integral and a series. Luckily enough for us we actually can

interchange the integral and the series in this case. Doing this gives and factoring the constant,  $B_n$ , out of the integral gives,

$$\begin{aligned}\int_{-L}^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx &= \sum_{n=1}^{\infty} \int_{-L}^L B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx \\ &= \sum_{n=1}^{\infty} B_n \int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx\end{aligned}$$

Note that from example 2 above we have  $\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} L & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$

i.e.  $\left\{ \sin\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty}$  is orthogonal on  $-L \leq x \leq L$ .

So, what does this mean for us? As we work through the various values of  $n$  in the series and compute the value of the integrals all but one of the integrals will be zero. The only non-zero integral will come when we have  $n = m$ , in which case the integral has the value of  $L$ . Therefore, the only non-zero term in the series will come when we have  $n = m$  and our equation becomes,

$$\int_{-L}^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = B_m L$$

Finally all we need to do is dividing by  $L$  and we know have an equation for each of the coefficient

$$B_m = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx \quad m = 1, 2, 3, \dots$$

Next, note that because we're integrating two odd functions the integrand of this integral is even and so we also know that,

$$B_m = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx \quad m = 1, 2, 3, \dots$$

Summarizing all this work up the Fourier sine series of an odd function  $f(x)$  on  $-L \leq x \leq L$  is given by,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \quad , \text{ Where } B_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad n = 1, 2, 3, \dots$$

**Example1:** Find the Fourier sine series for  $f(x) = x$  on  $-L \leq x \leq L$ .

Solution: First note that the function we're working with is in fact an odd function and so this is something we can do. There really isn't much to do here other than to compute the coefficients for  $f(x) = x$ .

Here is that work and note that we're going to leave the integration by parts details to you to verify. Don't forget that  $n$ ,  $L$ , and  $\pi$  are constants!

$$\begin{aligned} B_n &= \frac{1}{L} \int_{-L}^L x \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2}{L} \left( \frac{L}{n^2 \pi^2} \right) \left( L \sin\left(\frac{n\pi x}{L}\right) - n\pi x \cos\left(\frac{n\pi x}{L}\right) \right) \Big|_0^L \\ &= \frac{2}{n^2 \pi^2} (L \sin(n\pi) - n\pi L \cos(n\pi)) \end{aligned}$$

These integrals can, on occasion, be somewhat messy especially when we use a general  $L$  for the endpoints of the interval instead of a specific number.

Now, taking advantage of the fact that  $n$  is an integer we know that

$\sin(n\pi) = 0$  and  $\cos(n\pi) = (-1)^n$ . We therefore have,

$$B_n = \frac{2}{n^2 \pi^2} (-n\pi L (-1)^n) = \frac{(-1)^{n+1} 2L}{n\pi} \quad n = 1, 2, 3, 4, \dots$$

The Fourier sine series is then,

$$x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right)$$

At this point we should probably point out that we will be doing most, if not all, of our work here on a general interval ( $-L \leq x \leq L$  or  $0 \leq x \leq L$ ) instead of intervals with specific numbers for the endpoints. There are a couple of reasons for this. First, it gives a much more general formula that will work for any interval of that form which is always nice. Secondly, when we run into this kind of work in the next chapter it will also be on general intervals so we may as well get used to them now.

Now, finding the Fourier sine series of an odd function is fine and good but what if, for some reason, we wanted to find the Fourier sine series for a function that is not odd? To see how to do this we're going to have to make a change. The above work was done on the interval  $-L \leq x \leq L$ . In the case of a function that is not odd we'll be working on the interval  $0 \leq x \leq L$ . The reason for this will be made apparent in a bit.

So, we are now going to do is to try to find a series representation for  $f(x)$  on the interval  $0 \leq x \leq L$  that is in the form,

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

or in other words,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

As we did with the Fourier sine series on  $-L \leq x \leq L$  we are going to assume that the series will in fact converge to  $f(x)$  and we will hold off discussing the convergence of the series in another section not on this module.

There are two methods of generating formulas for the coefficients,  $B_n$  although we'll see in a bit that they really the same way, just looked at from different perspectives.

The first method is to just ignore the fact that  $f(x)$  is odd and proceed in the same manner that we did above only this time we'll take advantage of the fact that we left as an exercise in the

previous section that  $\left\{ \sin\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty}$  also forms an orthogonal set on  $0 \leq x \leq L$  and that,

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} \frac{L}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

So, if we do this then all we need to do is multiply both sides of our series by  $\sin\left(\frac{m\pi x}{L}\right)$ , integral from 0 to  $L$  and interchange the integral and series to get,

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

Now, plugging in for the integral we arrive at,

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = B_m \left(\frac{L}{2}\right)$$

Upon solving for the coefficient we arrive at,  $B_m = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx \quad m = 1, 2, 3, 4, \dots$

Note that this is identical to the second form of the coefficients that we arrived at above by assuming  $f(x)$  was odd and working on the interval  $-L \leq x \leq L$ . The fact that we arrived at essentially the same coefficients is not actually all that surprising as we'll see once we have looked the second method of generating the coefficients.

Before we look at the second method of generating the coefficients we need to take a brief look at another concept. Given a function,  $f(x)$ , we define the *odd extension* of  $f(x)$  to be the new function,

$$g(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq L \\ f(-x) & \text{if } -L \leq x \leq 0 \end{cases}$$

It's pretty easy to see that this is an odd function.

$$g(-x) = -f(-(-x)) = -f(x) = -g(x) \quad \text{for } 0 < x < L$$

and we can also know that on  $0 \leq x \leq L$  we have that  $g(x) = f(x)$ . Also note that if  $f(x)$  is already an odd function then we in fact get  $g(x) = f(x)$  on  $-L \leq x \leq L$ .

**Example 2:** Sketch the odd extension of the given function.

$$f(x) = L - x \quad \text{on } 0 \leq x \leq L$$

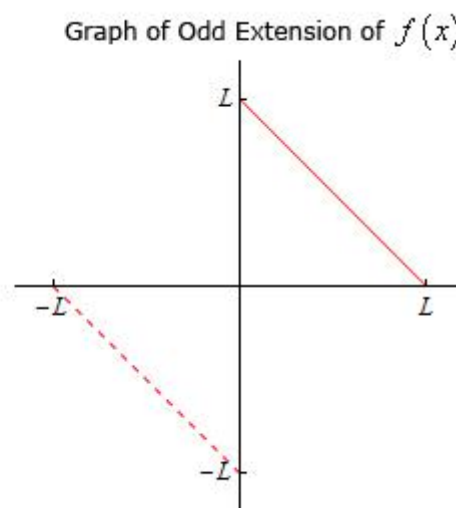
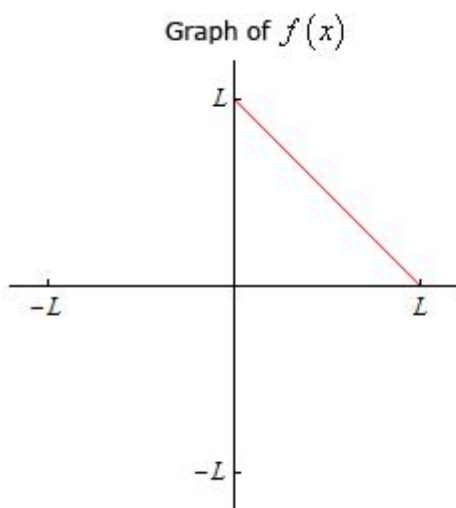
Solution: Not much to do with this other than to define the odd extension and then sketch it

Here is the odd extension of this function.

$$g(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq L \\ f(-x) & \text{if } -L \leq x \leq 0 \end{cases}$$

$$= \begin{cases} L - x & \text{if } 0 \leq x \leq L \\ -L - x & \text{if } -L \leq x \leq 0 \end{cases}$$

Below is the graph of both the function and its odd extension. Note that we've put the "extension" in with a dashed line to make it clear the portion of the function that is being added to allow us to get the odd extension.



**Example 3:** Find the Fourier sine series for  $f(x) = L - x$  on  $0 \leq x \leq L$

**Solution:** There really isn't much to do here other than computing the coefficients so here they are,

$$\begin{aligned} B_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^L (L - x) \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2}{L} \left( -\frac{L}{n^2 \pi^2} \right) \left[ L \sin\left(\frac{n\pi x}{L}\right) - n\pi(x - L) \cos\left(\frac{n\pi x}{L}\right) \right]_0^L \\ &= \frac{2}{L} \left[ \frac{L^2}{n^2 \pi^2} (n\pi - \sin n\pi) \right] \\ &= \frac{2L}{n\pi} (\text{why ?}) \end{aligned}$$

In the simplification process do not forget that  $n$  is an integer.

So, with the coefficients we get the following Fourier sine series for this function.

$$f(x) = \sum_{n=1}^{\infty} \frac{2L}{n\pi} \sin\left(\frac{n\pi x}{L}\right)$$

**Example 4:** Find the Fourier sine series for  $f(x) = \pi - x$  on  $0 \leq x \leq \pi$

$$\text{Answer: } f(x) = \sum_{n=1}^{\infty} \frac{2}{n} \sin(nx)$$

### 3.4.2 Fourier Cosine Series

In this section we're going to take a look at Fourier cosine series. We will start off much as we did in the previous section where we looked at Fourier sine series. Let's start by assuming that the function,  $f(x)$  we will be working with initially an even function.

[i.e.  $f(-x) = f(x)$  ] and we want to write a series representation for this function on  $-L \leq x \leq L$  in terms of cosines (which are also even). In other words we are going to look for the following,

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

This series is called a *Fourier cosine series* and note that in this case (unlike with Fourier sine series) we are able to start the series representation at  $n = 0$  since that term will not be zero as it was with sines. Also, as with Fourier Sine series, the argument of  $\frac{n\pi x}{L}$  in the cosines is being used only because it is the argument that we will be running into in the next topic. The only real requirement here is that the given set of functions we're using be orthogonal on the interval we're working on.

Note as well that we are assuming that the series will in fact converge to  $f(x)$  on  $-L \leq x \leq L$  at this point.

So, to determine a formula for the coefficients,  $A_n$ , we'll use the fact that  $\left\{ \cos\left(\frac{n\pi x}{L}\right) \right\}_{n=0}^{\infty}$  do form an orthogonal set on the interval  $-L \leq x \leq L$  as we showed in the previous section. In that section we also derived the following formula that we will need in a bit.

$$\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 2L & \text{if } n = m = 0 \\ L & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

We'll get a formula for the coefficients in almost exactly the same fashion that we did in the previous section. We'll start with the representation above and multiply both sides by

$\cos\left(\frac{m\pi x}{L}\right)$  where  $m$  is a fixed integer in the range  $\{1, 2, 3, 4, \dots\}$ . Doing this gives,

$$f(x) \cos\left(\frac{m\pi x}{L}\right) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right)$$

Next, we integrate both sides from  $x = -L$  to  $x = L$  and as we were able to do with the Fourier Sine series we can again interchange the integral and the series.

$$\int_{-L}^L f(x) \cos\left(\frac{m\pi x}{L}\right) dx = \int_{-L}^L \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx$$

$$= \sum_{n=0}^{\infty} A_n \int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx$$

We now know that the all of the integrals on the right side will be zero except when  $n = m$  because the set of cosines form an orthogonal set on the interval

$-L \leq x \leq L$  However, we need to be careful about the value of  $m$  (or  $n$  depending on the letter you want to use). So, after evaluating all of the integrals we arrive at the following set of formulas for the coefficients.

$$\text{For } m = 0, \int_{-L}^L f(x) dx = A_0(2L) \quad \Rightarrow \quad A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$$

$$\text{For } m \neq 0, \int_{-L}^L f(x) \cos\left(\frac{m\pi x}{L}\right) dx = A_m L \quad \Rightarrow \quad A_m = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{m\pi x}{L}\right) dx$$

Summarizing everything up then, the Fourier cosine series of an even function,  $f(x)$  on  $-L \leq x \leq L$  is given by,

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \quad \text{Where } A_n = \begin{cases} \frac{1}{2L} \int_{-L}^L f(x) dx & n = 0 \\ \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx & n \neq 0 \end{cases}$$

**Example 1:** Find the Fourier cosine series for  $f(x) = x^2$  on  $-L \leq x \leq L$

Solution: We clearly have an even function here and so all we really need to do is compute the coefficients and they are liable to be a little messy because we'll need to do integration by parts twice. We'll leave most of the actual integration details to you to verify.

$$A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{L} \int_0^L f(x) dx$$

$$= \frac{1}{L} \int_0^L x^2 dx = \frac{L^2}{3}$$

$$\begin{aligned}
\text{Again, } A_n &= \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \\
&= \frac{2}{L} \int_0^L x^2 \cos\left(\frac{n\pi x}{L}\right) dx \\
&= \frac{2}{L} \left( \frac{L}{n^3 \pi^3} \right) \left( 2 \ln \pi x \cos\left(\frac{n\pi x}{L}\right) + (n^2 \pi^2 x^2 - 2L^2) \sin\left(\frac{n\pi x}{L}\right) \right) \Big|_0^L
\end{aligned}$$

Substituting the values we get,

$$A_n = \frac{4L^2(-1)^n}{n^2 \pi^2} \quad n = 1, 2, 3, 4, \dots$$

The coefficients are then,  $A_0 = \frac{L^2}{3}$  and  $A_n = \frac{4L^2(-1)^n}{n^2 \pi^2}$   $n = 1, 2, 3, 4, \dots$

The Fourier cosine series is then,

$$\begin{aligned}
x^2 &= \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \\
&= \frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2(-1)^n}{n^2 \pi^2} \cos\left(\frac{n\pi x}{L}\right)
\end{aligned}$$

Note that we will often strip out  $n = 0$  from the series as we have done here because it will almost always be different from the other coefficients and it allows us to actually plug the coefficients into the series.

Now, just as we did in the previous section let's ask what we need to do in order to find the Fourier cosine series of a function that is not even. As with Fourier sine series when we make this change we'll need to move onto the interval  $0 \leq x \leq L$  now instead of  $-L \leq x \leq L$  and again we'll assume that the series will converge to  $f(x)$  at this point.

We could go through the work to find the coefficients here twice as we did with Fourier sine series; however there's no real reason to. So, while we could redo all the work above to get formulas for the coefficients let's instead go straight to the second method of finding the coefficients.

In this case, before we actually proceed with this we will need to define the even extension of a function,  $f(x)$  on  $-L \leq x \leq L$ . So; given a function  $f(x)$  we will define the even extension of the function as,

$$g(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq L \\ f(-x) & \text{if } -L \leq x \leq 0 \end{cases}$$

Showing that this is an even function is simple enough.

$g(-x) = f(-(-x)) = f(x) = g(x)$  for  $0 < x < L$  and we can see that  $g(x) = f(x)$  on  $0 \leq x \leq L$  and if  $f(x)$  is already an even function we get  $f(x) = g(x)$  on  $-L \leq x \leq L$ .

**Example 2:** Sketch the even extension of the given function.

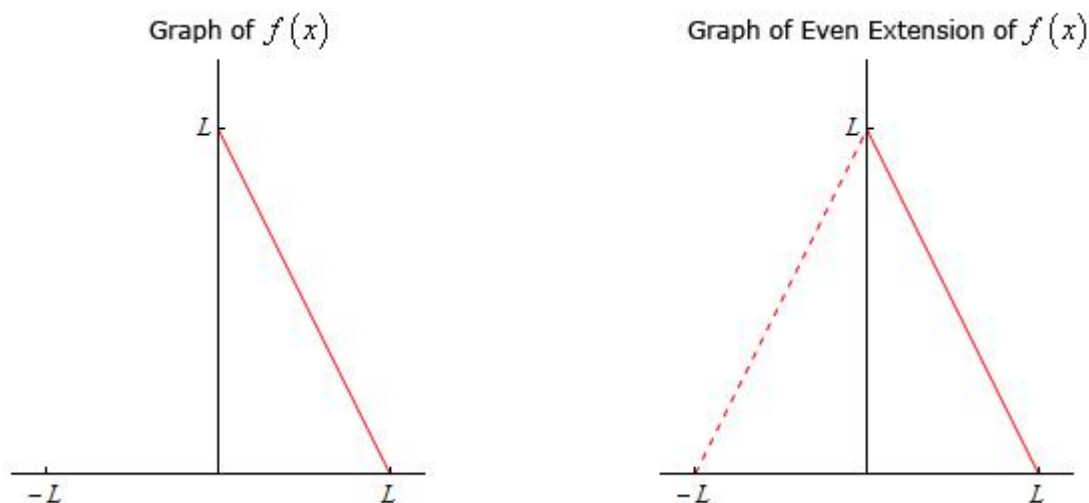
$$f(x) = L - x \text{ on } 0 \leq x \leq L$$

Solution: Here is the even extension of this function.

$$g(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq L \\ f(-x) & \text{if } -L \leq x \leq 0 \end{cases}$$

$$g(x) = \begin{cases} L - x & \text{on } 0 \leq x \leq L \\ L + x & \text{on } -L \leq x \leq 0 \end{cases}$$

Here is the graph of both the original function and its even extension. Note that we've put the "extension" in with a dashed line to make it clear the portion of the function that is being added to allow us to get the even extension



**Example 3:** Find the Fourier cosine series for  $f(x) = L - x$  on  $0 \leq x \leq L$

Solution: All we need to do is compute the coefficients so here is the work for that,

$$A_0 = \frac{1}{L} \int_0^L f(x) dx = \frac{1}{L} \int_0^L (L - x) dx = \frac{L}{2}$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^L (L - x) \cos\left(\frac{n\pi x}{L}\right) dx \text{ using integration by parts we}$$

arrive at,

$$A_n = \frac{2L}{n^2 \pi^2} (1 + (-1)^{n+1}) \quad n = 1, 2, 3, 4, \dots$$

The Fourier cosine series is then,

$$f(x) = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L}{n^2 \pi^2} (1 + (-1)^{n+1}) \cos\left(\frac{n\pi x}{L}\right)$$

Note that as we did with the first example in this section we stripped out the  $A_0$  term before we plugged in the coefficients.

**Example 4:** Find the Fourier cosine series for  $f(x) = 2\pi - x$  on  $0 \leq x \leq 2\pi$

Let's leave this as an exercise for the reader.

### 3.5 Fourier series

In the previous two sections we have looked at Fourier sine and Fourier cosine series. It is now time to look at a Fourier series. With a Fourier series we are going to try to write a series representation for  $f(x)$  on  $-L \leq x \leq L$  in the form,

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

So, a Fourier series is, in some way a combination of the Fourier sine and Fourier cosine series. Also, like the Fourier sine/cosine series we'll not worry about whether or not the series will actually converge to  $f(x)$  or not at this point. For now we will just assume that it will converge (on this module we do not cover about the convergence of Fourier series)

Determining formulas for the coefficients,  $A_n$  and  $B_n$ , will be done in exactly the same manner as we did in the previous two sections. We will take advantage of the fact that

$\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$  and  $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$  are mutually orthogonal on  $-L \leq x \leq L$  as we agreed

previously. We will also need the following formulas that we derived when we proved the two sets were mutually orthogonal.

$$\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 2L & \text{if } n = m = 0 \\ L & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} L & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0$$

So, let's start off by multiplying both sides of the series above by  $\cos\left(\frac{m\pi x}{L}\right)$  and integrate both sides from  $-L$  to  $L$  and interchange the integral and summation to get,

$$\int_{-L}^L f(x) \cos\left(\frac{m\pi x}{L}\right) dx = \sum_{n=0}^{\infty} A_n \int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx + \sum_{n=1}^{\infty} B_n \int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx$$

We can now take advantage of the fact that the sines and cosines are mutually orthogonal and also using the above facts we arrive at

$$A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx \quad A_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \quad m = 1, 2, 3, 4, \dots$$

Again we get,

$$B_m = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx \quad m = 1, 2, 3, 4, \dots$$

In the previous two sections we also took advantage of the fact that the integrand was even to give a second form of the coefficients in terms of an integral from 0 to  $L$ . However, in this case we don't know anything about whether  $f(x)$  will be even, odd, or more likely neither even nor odd. Therefore, this is the only form of the coefficients for the Fourier series.

**Example1:** Find the Fourier series for  $f(x) = \begin{cases} L & \text{if } -L \leq x \leq 0 \\ 2x & \text{if } 0 \leq x \leq L \end{cases}$  on  $-L \leq x \leq L$

Solution: Because of the piece-wise nature of the function the work for the coefficients is going to be a little unpleasant but let's get on with it.

$$\begin{aligned}
A_0 &= \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{2L} \left[ \int_{-L}^0 f(x) dx + \int_0^L f(x) dx \right] \\
&= \frac{1}{2L} \left[ \int_{-L}^0 L dx + \int_0^L 2x dx \right] \\
&= \frac{1}{2L} [L^2 + L^2] \\
&= L
\end{aligned}$$

$$\begin{aligned}
A_n &= \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L} \left[ \int_{-L}^0 f(x) \cos\left(\frac{n\pi x}{L}\right) dx + \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \right] \\
&= \frac{1}{L} \left[ \int_{-L}^0 L \cos\left(\frac{n\pi x}{L}\right) dx + \int_0^L 2x \cos\left(\frac{n\pi x}{L}\right) dx \right]
\end{aligned}$$

At this point it will probably be easier to do each of these individually.

$$\int_{-L}^0 L \cos\left(\frac{n\pi x}{L}\right) dx = \left( \frac{L^2}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \right) \Big|_{-L}^0 = \frac{L^2}{n\pi} \sin(n\pi) = 0$$

Again,

$$\begin{aligned}
\int_0^L 2x \cos\left(\frac{n\pi x}{L}\right) dx &= \left( \left( \frac{2L}{n^2 \pi^2} \right) \left( L \cos\left(\frac{n\pi x}{L}\right) + n\pi x \sin\left(\frac{n\pi x}{L}\right) \right) \right) \Big|_0^L \\
&= \frac{2L^2}{n^2 \pi^2} ((-1)^n - 1)
\end{aligned}$$

So, if we put all of this together we have,

$$A_n = \frac{2L}{n^2 \pi^2} ((-1)^n - 1), \quad n = 1, 2, 3, 4, \dots$$

So, we have gotten the coefficients for the cosines taken care of and now we need to take care of the coefficients for the sines.

$$\begin{aligned} B_n &= \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L} \left[ \int_{-L}^0 f(x) \sin\left(\frac{n\pi x}{L}\right) dx + \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \right] \\ &= \frac{1}{L} \left[ \int_{-L}^0 L \sin\left(\frac{n\pi x}{L}\right) dx + \int_0^L 2x \sin\left(\frac{n\pi x}{L}\right) dx \right]. \end{aligned}$$

As the coefficients for the cosines will probably be easier to do each of these individually.

$$\int_{-L}^0 L \sin\left(\frac{n\pi x}{L}\right) dx = \left( -\frac{L^2}{n\pi} \cos\left(\frac{n\pi x}{L}\right) \right) \Big|_{-L}^0 = \frac{L^2}{n\pi} (-1 + \cos(n\pi)) = \frac{L^2}{n\pi} ((-1)^n - 1)$$

Again,

$$\begin{aligned} \int_0^L 2x \sin\left(\frac{n\pi x}{L}\right) dx &= \left( \left( \frac{2L}{n^2\pi^2} \right) \left( L \sin\left(\frac{n\pi x}{L}\right) - n\pi x \cos\left(\frac{n\pi x}{L}\right) \right) \right) \Big|_0^L \\ &= \left( \frac{2L}{n^2\pi^2} \right) (L \sin(n\pi) - n\pi \cos(n\pi)) \\ &= \left( \frac{2L^2}{n^2\pi^2} \right) (-n\pi(-1)^n) \\ &= -\frac{2L^2}{n\pi} (-1)^n \end{aligned}$$

So, if we put all of this together we have,

$$B_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L} \left[ \frac{L^2}{n\pi} ((-1)^n - 1) - \frac{2L^2}{n\pi} (-1)^n \right]$$

$$= -\frac{L}{n\pi} (1 + (-1)^n) \quad n = 1, 2, 3, 4, \dots$$

So, after all that work the Fourier series is,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \\ &= A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \\ &= L + \sum_{n=1}^{\infty} \frac{2L}{n^2 \pi^2} ((-1)^n - 1) \cos\left(\frac{n\pi x}{L}\right) - \sum_{n=1}^{\infty} \frac{L}{n\pi} (1 + (-1)^n) \sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

**Exercise:** Find the Fourier series for  $f(x) = \begin{cases} L & \text{if } -L \leq x \leq 0 \\ 2x & \text{if } 0 \leq x \leq L \end{cases}$  on  $-\pi \leq x \leq \pi$

**Example 2:** Find the Fourier series for  $f(x) = x$  on  $-L \leq x \leq L$ .

**Solution:** Let's start with the integrals for  $A_n$ ,

$$A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{2L} \int_{-L}^L x dx = 0$$

$$A_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L} \int_{-L}^L x \cos\left(\frac{n\pi x}{L}\right) dx = 0$$

In both cases note that we are integrating an odd function ( $x$  is odd and cosine is even so the product is odd) over the interval  $[-L, L]$  and so we know that both of these integrals will be zero.

Next here is the integral for  $B_n$ ,

$$B_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L} \int_{-L}^L x \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx$$

In this case we're integrating an even function ( $x$  and sine are both odd so the product is even) on the interval  $[-L, L]$  and so we can "simplify" the integral as shown above. The reason for doing this here is not actually to simplify the integral however. It is instead done so that we can note that we did this integral back in the Fourier sine series section and so don't need to redo it in this section. Using the previous result we get,

$$B_n = \frac{(-1)^{n+1} 2L}{n\pi} \quad n = 1, 2, 3, 4, \dots$$

In this case the Fourier series is,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

**Exercise:** Find the Fourier series for  $f(x) = x$  on  $-2 \leq x \leq 2$ .

**Direction:** You can refer unit three of “**Applied three to be submitted...**” for this chapter (chapter one)!!!.